

CONTACT FLOWS AND INTEGRABLE SYSTEMS

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ABSTRACT. We consider Hamiltonian systems restricted to the hypersurfaces of contact type and obtain a partial version of the Arnold-Liouville theorem: the system not need to be integrable on the whole phase space, while the invariant hypersurface is foliated on invariant Lagrangian tori. In the second part of the paper we consider contact systems with constraints. As an example, it is shown that the well known Reeb flows on Brieskorn manifolds are contact integrable in a non-commutative sense.

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1. Introduction

Usually, integrable systems are considered within a framework of symplectic or Poisson geometry, but there is a well defined non-Hamiltonian setting as well (e.g., see [6, 35]). Among these uncommon integrable theories is a contact integrability studied in [3, 31, 21, 20, 18, 9]. The aim of this paper is to stress some natural applications of contact integrability to the Hamiltonian systems and to provide examples of contact integrable flows.

1.1. Contact flows and the Jacobi bracket. We start with some basic definitions in contact geometry. A *contact form* α on a $(2n + 1)$ -dimensional manifold M is a Pfaffian form satisfying $\alpha \wedge (d\alpha)^n \neq 0$. By a *contact manifold* (M, \mathcal{H}) we mean a connected $(2n + 1)$ -dimensional manifold M equipped with a nonintegrable *contact* (or *horizontal*) *distribution* \mathcal{H} , locally defined by a contact form: $\mathcal{H}|_U = \ker \alpha|_U$, U is an open set in M [22]. Two contact forms α and α' define the same contact distribution \mathcal{H} on U if and only if $\alpha' = a\alpha$ for some nowhere vanishing function a on U .

The condition $\alpha \wedge (d\alpha)^n \neq 0$ implies that the form $d\alpha|_x$ is nondegenerate (symplectic) structure restricted to \mathcal{H}_x . The conformal class of $d\alpha|_x$ is invariant under the change $\alpha' = a\alpha$. If \mathcal{V} is a linear subspace of \mathcal{H}_x , then we have well defined its orthogonal complement in \mathcal{H}_x with respect to $d\alpha|_x$, as well as the notion of the *isotropic* ($\mathcal{V} \subset \text{orth}_{\mathcal{H}} \mathcal{V}$), *coisotropic* ($\mathcal{V} \supset \text{orth}_{\mathcal{H}} \mathcal{V}$) and the *Lagrange* subspaces ($\mathcal{V} = \text{orth}_{\mathcal{H}} \mathcal{V}$) of \mathcal{H}_x . A submanifold $N \subset M$ is *pre-isotropic* if it transversal to \mathcal{H} and if $\mathcal{G}_x = T_x N \cap \mathcal{H}_x$ is an isotropic subspace of \mathcal{H}_x , $x \in N$. The last condition is equivalent to the condition that distribution $\mathcal{G} = \bigcup_x \mathcal{G}_x$ defines a foliation. It is *pre-Legendrian* submanifold if it is of maximal dimension $n + 1$, that is \mathcal{G} is a Lagrangian subbundle of \mathcal{H} .

A *contact diffeomorphism* between contact manifolds (M, \mathcal{H}) and (M', \mathcal{H}') is a diffeomorphism $\phi : M \rightarrow M'$ such that $\phi_* \mathcal{H} = \mathcal{H}'$. An *infinitesimal automorphism* of a contact structure (M, \mathcal{H}) is a vector field X , called a *contact vector field* such that its local 1-parameter group is made of contact diffeomorphisms. Locally, if $\mathcal{H} = \ker \alpha$, then $\mathcal{L}_X \alpha = \lambda \alpha$, for some smooth function λ .

From now on, we consider *co-oriented* (or *strictly*) *contact manifolds* (M, α) , where contact distributions \mathcal{H} are associated to globally defined contact forms α . The *Reeb vector field* Z is a vector field uniquely defined by

$$(1.1) \quad i_Z \alpha = 1, \quad i_Z d\alpha = 0.$$

The tangent bundle TM and the cotangent bundle T^*M are decomposed into $TM = \mathcal{Z} \oplus \mathcal{H}$ and $T^*M = \mathcal{H}^0 \oplus \mathcal{Z}^0$, where $\mathcal{Z} = \mathbb{R}Z$ is the kernel of $d\alpha$, \mathcal{Z}^0 and $\mathcal{H}^0 = \mathbb{R}\alpha$ are the annihilators of \mathcal{Z} and \mathcal{H} , respectively. The sections of \mathcal{Z}^0 are called *semi-basic forms*. Whence, we have decompositions of vector fields and 1-forms

$$(1.2) \quad X = (i_X \alpha)Z + \hat{X}, \quad \eta = (i_Z \eta)\alpha + \hat{\eta},$$

where \hat{X} is horizontal and $\hat{\eta}$ is semi-basic.

The mapping $\alpha^\flat : X \mapsto -i_X d\alpha$ carries X onto a semi-basic form. The form $d\alpha$ is non-degenerate on \mathcal{H} and the restriction of α^\flat to horizontal vector fields is an isomorphism whose inverse will be denoted by α^\sharp . The vector field

$$(1.3) \quad Y_f = fZ + \hat{Y}_f, \quad \hat{Y}_f = \alpha^\sharp(\hat{d}f).$$

is a contact vector field ($\mathcal{L}_{Y_f}\alpha = Z(f)\alpha$) and

$$(1.4) \quad \dot{x} = Y_f$$

is called *contact Hamiltonian equation* corresponding to f . Notice that $Z = Y_1$ and $f = i_{Y_f}\alpha$.

The mapping $f \mapsto Y_f$ is a Lie algebra isomorphism ($Y_{[f,g]} = [Y_f, Y_g]$) between the set $C^\infty(M)$ of smooth functions on M and the set \mathcal{N} of infinitesimal contact automorphisms. Here, the *Jacobi bracket* $[\cdot, \cdot]$ on $C^\infty(M)$ is defined by (see [22])

$$(1.5) \quad [f, g] = d\alpha(Y_f, Y_g) + fZ(g) - gZ(f) = Y_f(g) - gZ(f).$$

The Jacobi bracket does not satisfy the Leibniz rule. However, within a class of functions that are integrals of the Reeb flow, $C_\alpha^\infty(M) = \{f \in C^\infty(M) \mid Z(f) = [1, f] = 0\}$, the Jacobi bracket have properties as the usual Poisson brackets: g is a integral of (1.4) if and only if $[g, f] = 0$ and if g_1 and g_2 are two integrals of (1.4), then $[g_1, g_2]$ is also a integral.

1.2. Contact integrability. A Hamiltonian system on $2n$ -dimensional symplectic manifold P is *noncommutatively integrable* if it has $2n-r$ almost everywhere independent integrals $F_1, F_2, \dots, F_{2n-r}$ and F_1, \dots, F_r commute with all integrals

$$(1.6) \quad \{F_i, F_j\} = 0, \quad j = 1, \dots, r, \quad i = 1, \dots, 2n-r.$$

Regular compact connected invariant manifolds of the system are isotropic tori, generated by Hamiltonian flows of F_1, \dots, F_r . In a neighborhood of a regular torus, there exist canonical *generalized action-angle coordinates* such that integrals F_i , $i = 1, \dots, r$ depend only on actions and the flow is translation in angle coordinates (see Nehororshev [27] and Mishchenko and Fomenko [24]). If $r = n$ we have the usual commutative integrability described in the Arnold-Liouville theorem [2].

Noncommutative integrability of contact flows can be stated by the following theorem.

THEOREM 1.1. [18] *Suppose we have a collection of integrals $f_1, f_2, \dots, f_{2n-r}$ of equation (1.4) on a $(2n+1)$ -dimensional co-oriented contact manifold (M, α) , where $f = f_1$ or $f = 1$ and:*

$$(1.7) \quad [1, f_i] = 0, \quad [f_i, f_j] = 0, \quad i = 1, \dots, 2n-r, \quad j = 1, \dots, r.$$

Let T be a compact connected component of the level set $\{f_1 = c_1, \dots, f_{2n-r} = c_{2n-r}\}$ and $df_1 \wedge \dots \wedge df_{2n-r} \neq 0$ on T . Then T is diffeomorphic to a pre-isotropic $r+1$ -dimensional torus \mathbb{T}^{r+1} . There exist a neighborhood U of T and a diffeomorphism $\phi : U \rightarrow \mathbb{T}^{r+1} \times D$,

$$(1.8) \quad \phi(x) = (\theta, y, x) = (\theta_0, \theta_1, \dots, \theta_r, y_1, \dots, y_r, x_1, \dots, x_{2s}), \quad s = n-r,$$

where $D \subset \mathbb{R}^{2n-r}$, such that

(i) α has the following canonical form

$$(1.9) \quad \alpha_0 = (\phi^{-1})^* \alpha = y_0 d\theta_0 + y_1 d\theta_1 + \cdots + y_r d\theta_r + g_1 dx_1 + \cdots + g_{2s} dx_{2s},$$

where y_0 is a smooth function of y and g_1, \dots, g_{2s} are functions of (y, x) ;

(ii) the flow of Y_f on invariant tori is quasi-periodic

$$(1.10) \quad (\theta_0, \theta_1, \dots, \theta_r) \mapsto (\theta_0 + t\omega_0, \theta_1 + t\omega_1, \dots, \theta_r + t\omega_r), \quad t \in \mathbb{R},$$

where frequencies $\omega_0, \dots, \omega_r$ depend only on y .

Note that we can define the notion of noncommutative integrability of contact equations $\dot{x} = X$ on contact manifolds (M, \mathcal{H}) without involving globally defined contact α (for more details see [18]).

The case $r = n$ corresponds to contact commutative integrability defined by Banyaga and Molino, where the invariant tori are pre-Legendrian [3] (see also [20, 9]). Slightly different notion of (commutative) contact integrability in terms of the *modified Poisson* (or the *Jacobi-Mayer*) *bracket* (\cdot, \cdot) [14],

$$(f, g) \alpha \wedge (d\alpha)^n = n df \wedge dg \wedge \alpha \wedge (d\alpha)^{n-1},$$

is given by Webster [31]. Note that the modified Poisson bracket satisfies the Leibniz rule, but does not satisfy the Jacobi identity.

1.3. Outline and results of the paper. In section 2 we consider integrable Hamiltonian systems restricted to the hypersurfaces of contact type and prove that the appropriate Reeb flows are examples of contact integrable systems (Theorem 2.1). This is a contact analogue of the construction of integrable geodesic flows by the use of the Maupertuis principle [7]. As an example, a contact algebraic hypersurface of degree 4 in $(\mathbb{R}^{2n+2}(q, p), dp \wedge dq)$ with a integrable Reeb flow associated to the Gefand–Cetin system on $u(n+1)$ is given (Proposition 2.1).

As a bi-product, in Section 3, we obtain a statement on isoenergetic integrability (Theorem 3.1, Corollary 3.1), where the system not need to be integrable on the whole phase space, but the relation (1.6) holds on a fixed isoenergetic hypersurface M of contact type. Then, as in the case of usual (non-commutative) integrability, M is foliated on invariant (isotropic) Lagrangian tori with quasi-periodic dynamics.

The construction of partial integrals for natural mechanical systems, on a fixed isoenergetic hypersurface, is studied by Birkhoff [4, 5]. Although the problem is classical and some variants of restricted integrability on invariant manifolds are already studied (see [29] and references therein), Theorem 3.1 is a benefit of a noncommutative contact integrability given in [18].

The situation is also closely related to a notion of systems of Hess-Appel’rot type [12, 17, 13], where the invariant manifold is foliated on Lagrangian tori, but not with quasi-periodic dynamics (see Corollary 3.2).

In Section 4 we consider contact flows with constraints and derive the contact version of Dirac’s construction for constrained Hamiltonian systems (Theorem 4.1). As an example, in Section 5 we start from a K -contact sphere S^{2n+1} (see [33]) with a contact integrable Reeb flow (Proposition 5.2). Then, in the rank 1 case, we change our contact manifold by imposing the constraints in the well known construction of the contact structure on Brieskorn manifolds [23, 10]. It is well known that all

trajectories of the corresponding Reeb flow are closed, and, therefore, the system is integrable in a noncommutative sense. Here, we describe the Jacobi bracket within the corresponding Lie algebra of integrals (Theorem 5.1).

Further, for a class of the Brieskorn manifolds diffeomorphic to the standard spheres S^{4m+1} , $m \in \mathbb{N}$, and contact flows studied by Ustilovsky [30], we prove contact noncommutative integrability of the flows, with generic invariant pre-isotropic tori of dimension $m + 1$ (Proposition 5.4).

Finally, we note that the integrability of the Reeb flow on a sphere S^{2n+1} is a particular case of the integrability of the Reeb flows on compact K -contact manifolds (Proposition 6.1).

2. The Maupertuis–Jacobi principle and integrable contact flows

2.1. Hypersurfaces of contact type. Let (P, ω) be a symplectic $2n$ dimensional manifold. Let H be a smooth function on P . Consider the Hamiltonian equation

$$(2.1) \quad \dot{x} = X_H,$$

where the Hamiltonian vector field X_H is defined by

$$(2.2) \quad i_{X_H} \omega(\cdot) = \omega(X_H, \cdot) = -dH(\cdot).$$

The Hamiltonian H is the first integral of the system. Let M be a regular connected component of the invariant variety $H = h$ ($dH|_M \neq 0$).

Since $dH(\xi) = 0$, $\xi \in T_x M$, from (2.2) we see that X_H generate the symplectic orthogonal of $T_x M$ for all $x \in M$ – the *characteristic line bundle* \mathcal{L}_M of M . It is exactly the kernel of the form ω restricted to M :

$$\mathcal{L}_M = \{\xi \in T_x M \mid \omega(\xi, T_x M) = 0, x \in M\}.$$

Note that \mathcal{L}_M is determined only by M and not by H . If F is another Hamiltonian defining M , $M \subset F^{-1}(c)$, $dF|_M \neq 0$, then the restriction of Hamiltonian vector fields X_H and X_F to M are proportional.

An orientable hypersurface M of a symplectic manifold (P, ω) is of *contact type* if there exist a 1-form α on M satisfying

$$d\alpha = j^* \omega, \quad \alpha(\xi) \neq 0, \xi \in \mathcal{L}_M, \xi \neq 0$$

where $j : M \rightarrow P$ is the inclusion, see Weinstein [32]. If (M, α) is of contact type, since $\mathcal{L} = \ker \omega_M$, the kernel of α

$$\mathcal{H} = \{\xi \in T_x M \mid \alpha(\xi) = 0, x \in M\}$$

is a $(2n - 2)$ -dimensional nonintegrable distribution on which $d\alpha = \omega$ is nondegenerate. Consequently, $\alpha \wedge d\alpha^{n-1}$ is a volume form on M and (M, \mathcal{H}) is a co-oriented contact manifold.

Note that, since the corresponding Reeb vector field (1.1) is a section of $\ker d\alpha$, it is proportional to $X_H|_M$. Therefore, the Reeb flow, up to a time reparametrization, coincides with the Hamiltonian flow restricted to M . In particular, a closed Reeb orbit is a closed orbit of the Hamiltonian flow, which was the motivation for introducing a concept of contact type hypersurfaces [32]. For example, by using

the contact homology, an estimate of the number of closed orbits for the restricted 3-body problem is recently given in [1].

2.2. Example. Let $(P, \omega = d\alpha)$ be an exact symplectic manifold and let M be a regular component of an isoenergetic surface $H^{-1}(h)$. If $\alpha(X_H)|_M \neq 0$ then M is of contact type with respect to α . Then α has no zeros in some open neighborhood of M .

Suppose that the 1-form α has no zeros in some open neighborhood of M . Then, from the nondegeneracy of ω , there exist a unique vector field E such that

$$(2.3) \quad i_E \omega = \alpha.$$

The vector field E has no zeros. From Cartan's formula, the condition $i_E \omega = \alpha$ is equivalent to $L_E \omega = \omega$, i.e., E is the *Liouville vector field* of ω .

A regular connected component M of an isoenergetic surface $H^{-1}(h)$ is of contact type with respect to α if and only if the Liouville vector field defined by (2.3) is transverse to M , i.e., $E(H)|_M \neq 0$ (e.g., see Libermann and Marle [22]). It can be shown that the Reeb vector field is given by

$$(2.4) \quad Z = X_H / E(H)|_M$$

(e.g., see [19]). For example, the Reeb vector field Z coincides with the restrictions to M of Hamiltonian vector fields of functions

$$(2.5) \quad H_0 = \frac{H - h}{E(H)} \quad (H_0|_M = 0),$$

$$(2.6) \quad H_{MJ} = \frac{E(H)}{4h - 4H + 2E(H)} \quad (H_{MJ}|_M = \frac{1}{2}).$$

The transformation $H \mapsto H_0$ is a variation of Moser's regularization of Kepler's problem [25, 19]. On the other hand, given a natural mechanical system $(Q, \langle \cdot, \cdot \rangle, V)$,

$$(2.7) \quad H(q, p) = \frac{1}{2} \langle p, p \rangle + V(q) = \frac{1}{2} \sum_{ij} K^{ij} p_i p_j + V(q),$$

and a isoenergetic surface $M_h = H^{-1}(h)$, $h > \max_Q V$, the function (2.6)

$$H_{MJ}(q, p) = \langle p, p \rangle / (4(h - V(q)))$$

is the Hamiltonian of the geodesic flow of the Maupertuis–Jacobi metric $ds_{MJ}^2 = 2(h - V(q))ds^2$ on Q . Here $\alpha = pdq$ is the canonical 1-form and $E = \sum_i p_i \partial / \partial p_i$ is the *standard Liouville vector field* on $T^*Q(q, p)$.

Furthermore, in the case of the natural mechanical systems, a compact regular component M of the isoenergetic hypersurface $H^{-1}(h)$ is always of compact type. It is clear when $h > \max_Q V$. In general, if $h < \max_Q V$, then we can perturb the canonical 1-form pdq by a closed 1-form β , such that M is of contact type with respect to $pdq + \beta$ (e.g., see [16]).

2.3. Integrability of the Reeb flow. It is well known that the standard metrics on a rotational surface and on an ellipsoid have the geodesic flows integrable by means of an integral polynomial in momenta of the first (Clairaut) and the second degree (Jacobi), respectively [2]. The natural question is the existence of metrics on a sphere S^2 with polynomial integral which can't be reduced to a linear or a quadratic one. The first examples, the Kovalevskaya ds_K^2 and Goryachev–Chaplygin ds_{GC}^2 metrics with additional integrals of 4-th and 3-th degrees, are given by Bolsinov, Kozlov and Fomenko (see [7]). Namely, the motion of a rigid body about a fixed point in the presence of the gravitation field admits $SO(2)$ –reduction (rotations about the direction of gravitational field). Taking the integrable Kovalevskaya and Goryachev–Chaplygin cases we get the integrable systems on T^*S^2 . The metrics ds_K^2 and ds_{GC}^2 are then the appropriate Maupertuis–Jacobi metrics on the sphere.

The following statement is a contact generalization of the construction given in [7].

Let $\{\cdot, \cdot\}$ be the canonical Poisson bracket on a symplectic manifold (P, ω) .

THEOREM 2.1. *Suppose that the Hamiltonian equation (2.1) are completely integrable in a noncommutative sense with respect to the integrals $F_1 = H, \dots, F_{2n-r}$ satisfying (1.6). Let $M = H^{-1}(h)$ be a contact type hypersurface, such that the restrictions $F_2|_M, \dots, F_{2n-r}|_M$ are independent. Then the Reeb flow on M is contact completely integrable in a noncommutative sense with respect to the integrals $F_2|_M, \dots, F_{2n-r}|_M$.*

PROOF. Let $j : M \rightarrow P$ be the inclusion and α be a contact form on M , such that $j^*\omega = d\alpha$. First note that $f_i = F_i \circ j$, $i = 2, \dots, 2n - r$ are integrals of the Reeb flow

$$\dot{x} = Z.$$

Indeed, since $X_H(F_i) = 0$ and the Reeb vector field Z is proportional to \tilde{X}_H ($j_*\tilde{X}_H = X_H$), we have $Z(f_i) = 0$, $i = 2, \dots, 2n - r$. That is, df_i are semi-basic forms on M , or in terms of the Jacobi bracket (1.5):

$$[1, f_i] = 0, \quad i = 2, \dots, 2n - r.$$

We shall prove $[f_k, f_i] = 0$, $i = 2, \dots, 2n - r$, $k = 2, \dots, r$. We have

$$\begin{aligned} (2.8) \quad [f_k, f_i] &= Y_{f_k}(f_i) - f_i Z(f_k) = Y_{f_k}(f_i) \\ &= (f_k Z + \alpha^\sharp(\widehat{df_k}))(f_i) = \alpha^\sharp(df_k)(f_i). \end{aligned}$$

Recall that $\hat{Y}_{f_k} = \alpha^\sharp(df_k)$ is a horizontal vector field ($\alpha(\hat{Y}_{f_k}) = 0$) defined by

$$(2.9) \quad i_{\hat{Y}_{f_k}} d\alpha = -df_k.$$

On the other hand, the Hamiltonian vector field X_{F_k} is tangent to M and satisfies

$$(2.10) \quad i_{X_{F_k}} \omega = -dF_k.$$

By taking the pull back of (2.10) and combining with (2.9) we get

$$\hat{Y}_{f_k} - \tilde{X}_{F_k} \subset \ker d\alpha, \quad j_*\tilde{X}_{F_k} = X_{F_k}|_M,$$

i.e.,

$$\hat{Y}_{f_k} = \tilde{X}_{F_k} + \rho_k Z,$$

for a certain function $\rho_k : M \rightarrow \mathbb{R}$.

Therefore, from (2.8) and (1.6) we obtain the required relations:

$$\begin{aligned} [f_k, f_i] &= (\tilde{X}_{F_k} + \rho_k Z)(f_i) = \tilde{X}_{F_k}(f_i) \\ &= j^* X_{F_k}(F_i) = j^* \{F_i, F_k\} = 0, \end{aligned}$$

for $i = 2, \dots, 2n - r$, $k = 2, \dots, r$. \square

2.4. Example. Below we give the example of an integrable contact flow, which is not the geodesic flow of a Maupertuis–Jacobi metric.

Consider the simplest (noncommutative) integrable system on the standard linear symplectic space $(\mathbb{R}^{2n+2}(q, p), dp \wedge dq)$: the system of $n + 1$ harmonic oscillators with the Hamiltonian

$$(2.11) \quad H_0 = \sum_{i=0}^n \frac{1}{2a_i} (q_i^2 + p_i^2),$$

where a_i are positive numbers.

If instead of the canonical 1-form pdq , we take

$$(2.12) \quad \alpha_0 = \sum_{i=0}^n p_i dq_i - \frac{1}{2} d\left(\sum_{i=0}^n p_i q_i\right) = \frac{1}{2} \sum_i p_i dq_i - q_i dp_i,$$

then $d\alpha_0 = d(pdq) = dp \wedge dq$ and the only zero of α_0 is at the origin 0. The corresponding Liouville vector field is

$$E = \frac{1}{2} \sum_i q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i}.$$

Let

$$E_h : \quad H_0 = \sum_{i=0}^n \frac{1}{2a_i} (q_i^2 + p_i^2) = h, \quad h > 0.$$

We have $E(H_0)|_{E_h} = h \neq 0$. Therefore, E_h is of contact type with respect to α_0 . According to (2.4), the Reeb vector field on (E_h, α_0) is given by

$$(2.13) \quad Z = \frac{1}{h} X_{H_0}.$$

The contact ellipsoid (E_h, α_0) is contactomorphic to the K -contact structure on a sphere S^{2n+1} [33] (see Section 4). In particular, for $a_0 = a_1 = \dots = a_n = 1$, we get the standard contact structure on the sphere $S^{2n+1} = H_0^{-1}(h)$, where the characteristic line bundle defines the Hopf fibration.

LEMMA 2.1. *The standard linear action of $U(n+1)$ on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ ($p + iq = z$) is Hamiltonian with the momentum mapping*

$$\Phi : \mathbb{R}^{2n+2} \rightarrow \mathfrak{u}(n+1) \cong \mathfrak{u}(n+1)^*, \quad \Phi(q, p) = p \wedge q + i(q \otimes q + p \oplus p).$$

Here $\mathfrak{u}(n+1)$ is identified with $\mathfrak{u}(n+1)^*$ by the use of the product proportional to the Killing form: $\langle \xi, \eta \rangle = -\frac{1}{2} \text{tr}(\xi\eta)$.

The Hamiltonian (2.11), for $a_0 = a_1 = \dots = a_n = 1$, is a collective function - the pull-back of a linear Cazimir function

$$K_0(\xi) = -\frac{1}{2} \text{tr}(\xi\zeta), \quad \zeta = \text{diag}(i, \dots, i)$$

via the momentum mapping Φ . Note that $\Phi(\mathbb{R}^{2n+2})$ is the union of singular adjoint orbits. For $(q, p) \neq 0$, the orbit through $\Phi(q, p)$ is diffeomorphic to a complex projective space $\mathbb{CP}^n \cong U(n+1)/U(n) \times U(1)$. On the other hand, the orbits of $U(n+1)$ -action on \mathbb{R}^{2n+2} for $(q, p) \neq 0$ are the spheres $H_0^{-1}(h)$.

Let us take an arbitrary integrable system on $\mathfrak{u}(n+1)$. For example, consider the Gelfand–Cetin system that is defined by the filtration of Lie algebras (e.g., see [15])

$$\mathfrak{u}(1) \subset \mathfrak{u}(2) \subset \dots \subset \mathfrak{u}(n+1).$$

Let $\xi_{\mathfrak{u}(k)}$ be the projection of ξ to $\mathfrak{u}(k)$ with respect to the scalar product $\langle \cdot, \cdot \rangle$. The Euler equation

$$\dot{\xi} = [\xi, \nabla K(\xi)]$$

with the Hamiltonian

$$K(\xi) = -\frac{1}{2} \sum_{k=0}^n \lambda_k \text{tr}(\xi_{\mathfrak{u}(k+1)} \xi_{\mathfrak{u}(k+1)})$$

is completely integrable. Note that the $U(n+1)$ -action is multiplicity free, so the system with collective Hamiltonian $H = K \circ \Phi$ is also integrable and we need only Noether integrals for complete integrability [15]. Indeed, we have

$$(2.14) \quad H = \sum_{k=0}^n \lambda_k H_k(q, p),$$

where

$$H_k = \frac{1}{2} \sum_{i=0}^k (q_i^2 + p_i^2)^2 + \sum_{0 \leq i < j \leq r} (q_i q_j + p_i p_j)^2 + (q_j p_i - p_j q_i)^2, \quad k = 0, \dots, n$$

are commuting integrals of the system. If parameters $\lambda_0, \dots, \lambda_r$ ($r < n$) are equal to zero, then the momentum mapping $\Phi_{\mathfrak{u}(r+1)}$ of the $U(r+1)$ -symmetry is conserved and the system is integrable in a noncommutative sense.

Let

$$\Sigma : \quad H = h \neq 0.$$

It is an algebraic surface of degree 4. Since $E(H) = 2h|_{\Sigma}$, Σ is of contact type and the Reeb flow is $Z = (2h)^{-1} X_H|_{\Sigma}$.

We can summarize the considerations above in the following statement.

PROPOSITION 2.1. *The Reeb flow on Σ is completely integrable.*

Similarly, let a compact connected Lie group G acts in a Hamiltonian way on a symplectic manifold (P, ω) with the equivariant momentum mapping $\Phi : P \rightarrow \mathfrak{g}^*$. Let $K : \mathfrak{g}^* \rightarrow \mathbb{R}$ be a Hamiltonian function such that the Euler equations

$$(2.15) \quad \dot{\mu} = -\text{ad}_{dK(\mu)}^* \mu$$

are completely integrable on general co-adjoint orbits $\mathcal{O}(\mu) \subset \Phi(M)$ with a set of Lie–Poisson commuting integrals f_i , $i = 1, \dots, N$, $N = \dim \mathcal{O}(\mu)$. Then the Hamiltonian equations on P with a collective Hamiltonian function $H = K \circ \Phi$ are completely integrable (in the non-commutative sense). The complete set of first integrals is

$$\{f_i \circ \Phi \mid i = 1, \dots, N\} + C_G^\infty(P),$$

where $C_G^\infty(P)$ is the algebra of G -invariant functions [8]. Thus, if a isoenergetic hypersurface $M = H^{-1}(h)$ is of a contact type, the associated Reeb vector field will be completely integrable.

3. Isoenergetic and partial integrability

We can say that Theorem 2.1 is something that one could expect. Now we formulate slightly unexpected statement.

THEOREM 3.1 (Isoenergetic integrability). *Let $M = H^{-1}(h)$ be a contact type hypersurface. Suppose a collection of functions $F_1 = H, \dots, F_{2n-r}$ satisfy (1.6) on M and that the restrictions $F_2|_M, \dots, F_{2n-r}|_M$ are independent. Then the Reeb flow on M is contact completely integrable in a noncommutative sense with respect to the integrals $F_2|_M, \dots, F_{2n-r}|_M$. The regular compact connected components of the invariant level sets*

$$(3.1) \quad H = F_1 = h, \quad F_2 = c_2, \quad \dots, \quad F_{2n-r} = c_{2n-r}$$

are invariant isotropic tori of (P, ω) (or pre-isotropic tori considered on (M, α)) and the dynamics $\dot{x} = X_H$ is proportional to the quasi-periodic dynamic of the Reeb flow on M .

PROOF. As in the proof of the usual noncommutative integrability, from

$$X_{F_i}(F_j) = \{F_j, F_i\} = 0|_M, \quad i = 1, \dots, r, \quad j = 1, \dots, 2n - r,$$

we have that X_{F_i} , $i = 1, \dots, r$ are tangent to the invariant sets (3.1). Also, if T is regular component of (3.1), from the dimensional reasons, its tangent space is spanned with X_{F_i} , $i = 1, \dots, r$. This is an isotropic manifold since

$$\omega(X_{F_i}, X_{F_j}) = \{F_j, F_i\} = 0|_M, \quad i, j = 1, \dots, r.$$

It is a nontrivial fact that T admits a transitive \mathbb{R}^r -action and therefore it is diffeomorphic to a r -dimensional torus. This, together with a noncommutative integrability of the Reeb flow, follows from the proof of Theorem 2.1, where we used the relations (1.6) only restricted to M . \square

In particular, for $n = r$ we have the isoenergetic version of the Arnold-Liouville theorem:

COROLLARY 3.1 (Isoenergetic Arnold-Liouville theorem). *Let $M = H^{-1}(h)$ be a contact type hypersurface. Suppose a collection of functions $F_1 = H, \dots, F_n$ Poisson commute on M*

$$(3.2) \quad \{F_i, F_j\} = 0|_M$$

and that the restrictions $F_2|_M, \dots, F_n|_M$ are independent. Then the Reeb flow on M is contact completely integrable with respect to the integrals $F_2|_M, \dots, F_n|_M$.

Thus, the system need not be integrable on the whole phase space, while the isoenergetic hypersurface M is foliated on Lagrangian tori, or pre-Legendrian in the contact sense. In a neighborhood of a torus, there exist contact action-angle coordinates $(\theta_0, \dots, \theta_{n-1}, y_1, \dots, y_{n-1})$, such that y_i depends on the integrals $F_2|_M, \dots, F_n|_M$, the contact form has the canonical form $y_0(y_1, \dots, y_{n-1})d\theta_0 + y_1d\theta_1 + \dots + y_{n-1}d\theta_{n-1}$, and in which the Reeb vector field is linearized

$$Z = \sum_{i=0}^{n-1} \omega_i \frac{\partial}{\partial \theta_i}, \quad \omega_0 = \left(-y_0 + \sum_{i=1}^{n-1} y_i \frac{\partial y_0}{\partial y_i} \right)^{-1}, \quad \omega_i = -\frac{1}{\omega_0} \frac{\partial y_0}{\partial y_i}.$$

Although the problem of existence of polynomial in momenta first integrals for natural mechanical systems, on a fixed isoenergetic hypersurface, is well known and goes back to Birkhoff [4, 5] (the examples are given by Yehia, see [34] and references therein), to the authors knowledge, the formulation of the isoenergetic integrability was not given yet. An another approach to the integrability on invariant manifolds is given by Nekhoroshev [28, 29].

Next, note that we only need to have relations (3.2) on a fixed hypersurface $M = F_1^{-1}(c)$ to conclude that M is foliated in invariant tori.

COROLLARY 3.2. *Suppose that a Hamiltonian system $\dot{x} = X_H$ has $n-1$ commuting integrals $F_1 = H, F_2, \dots, F_{n-1}$ and an invariant relation*

$$\Sigma : \quad F_0 = 0,$$

that is, the trajectories with initial conditions on Σ stay on Σ for all time t . If Σ is of the contact type manifold and if it is invariant for all Hamiltonian flows X_{F_i} , then the compact regular components of the invariant varieties

$$F_0 = 0, \quad H = F_1 = c_1, \quad F_2 = c_2, \quad \dots, \quad F_{n-1} = c_{n-1}$$

are Lagrangian tori.

The situation is closely related to a notion of a partial integrability and systems of Hess-Appel'rot type introduced by Dragović and Gajić [12] (see also [17, 13]).

3.1. Remark. If we are interested in the dynamics in the invariant neighborhood $U \subset M$ of the regular compact invariant level set (3.1) (with the property $dF_1 \wedge \dots \wedge dF_{2n-r} \neq 0|_U$), then instead of the condition that M is of contact type in Theorem 3.1, we can assume slightly weaker condition that U is of compact type. Then we have contact integrability of the Reeb flow restricted to U . Similarly, the invariant relation Σ in Corollary 3.2 does not need to be of contact type.

3.2. Example. A classical example of a partially integrable system is the Hess-Appel'rot case of the heavy rigid body motion around a fixed point. The phase space is the cotangent bundle of $SO(3)$. In Euler's angles (φ, θ, ψ) , the Hamiltonian of the system can be written in the form

$$H = \frac{1}{2}(aM_1^2 + aM_2^2 + bM_3^2 + 2cM_1M_3) + k \cos \theta, \quad M_3 = p_\varphi$$

$$M_1 = \frac{\sin \varphi}{\sin \theta}(p_\psi - p_\varphi \cos \theta) + p_\theta \cos \varphi, \quad M_2 = \frac{\cos \varphi}{\sin \theta}(p_\psi - p_\varphi \cos \theta) - p_\theta \sin \varphi.$$

The system has two integrals, the Hamiltonian function H and the Noether integral $M_z = p_\psi$ corresponding to the symmetry of the system with respect to the rotations around the vertical axes \vec{e}_z . Also, the system has the invariant relation

$$\Sigma : \quad M_3 = p_\varphi = 0.$$

Since the Hamiltonian flow of the function M_3 is given by the vector field $X_{M_3} = \partial/\partial\varphi$ (rotations of the body around the axes \vec{e}_3 fixed in the body), we see that Σ is not of the contact type with respect to the canonical 1-form $p_\varphi d\varphi + p_\theta d\theta + p_\psi d\psi$, but it is of the contact type with respect to the perturbation of the canonical 1-form by a closed 1-form $d\varphi$:

$$\alpha = p_\varphi d\varphi + p_\theta d\theta + p_\psi d\psi + d\varphi, \quad \alpha(X_{M_3}) \equiv 1.$$

Further, since $\{M_3, M_z\} = \{p_\varphi, p_\psi\} = 0$, from Corollary 3.2 we obtain that the regular connected invariant level sets

$$H = h, \quad M_z = c, \quad M_3 = 0,$$

are Lagrangian tori. This is well known and can be seen directly from the fact that, after confining to Σ , the Hamiltonian of the Hess-Appel'rot case coincides with the Hamiltonian of the integrable Lagrange case of the heavy rigid body motion. However, the dynamics on Σ of the Hess-Appel'rot system is quite different from the one of the Lagrange top.

4. Contact systems with constraints

A *contact submanifold* of the contact manifold (M, \mathcal{H}_M) is a triple (N, \mathcal{H}_N, j) , where (N, \mathcal{H}_N) is a contact manifold and $j : N \rightarrow M$ is an embedding such that $j_*^{-1}(\mathcal{H}_M) = \mathcal{H}_N$.

Let (M, α) be a co-oriented contact manifold and $j : N \rightarrow M$ an embedding. If we define

$$\mathcal{H}_N = \{X \in TN \mid j_*(X) \in \mathcal{H}_M\} = j_*^{-1}(\mathcal{H}_M),$$

then $\mathcal{H}_N = \ker(j^*\alpha)$. The distribution \mathcal{H}_N is of codimension one, if N is transverse to \mathcal{H}_M . In order to induce a contact structure on N it is also necessary that $dj^*\alpha$ is non-degenerate on $(\mathcal{H}_N)_x$, for all $x \in N$. To summarize, $(N, j^*\alpha)$ is a *contact co-oriented submanifold* of (M, α) , if N is transverse to \mathcal{H}_M and if $dj^*\alpha$ is non-degenerate on $\ker(j^*\alpha)$.

We derive a contact version of Dirac's construction which deals with constrained Hamiltonian vector fields on symplectic manifolds (e.g., see [26]).

THEOREM 4.1. *Let (M, α) be a $(2n+1)$ -dimensional co-oriented contact manifold, G_1, \dots, G_{2k} smooth functions on M ,*

$$(4.1) \quad N = \{x \in M \mid G_1(x) = \dots = G_{2k}(x) = 0\},$$

and $j : N \rightarrow M$ be the corresponding embedding.

(a) *If $[1, G_j] = 0|_N$, $j = 1, \dots, 2k$ and*

$$(4.2) \quad \det([G_j, G_i]) \neq 0|_N$$

then (N, j^α) is a contact submanifold of (M, α) with the Reeb vector field that is the restriction of the the Reeb vector field Z of (M, α) .*

(b) *Let f be a smooth function on M and*

$$W_f = Y_f - \sum_{i=1}^{2k} \lambda_i Y_{G_i}.$$

Then the system

$$(4.3) \quad dG_j(W_f) = Y_f(G_j) - \sum_i \lambda_i Y_{G_i}(G_j) = 0 \quad j = 1, \dots, 2k$$

has a unique solution $\lambda_1 = \lambda_1(f), \dots, \lambda_{2k} = \lambda_{2k}(f)$ on N . For the given multipliers,

$$Y_f^* = W_f$$

is the contact Hamiltonian vector field of the function f restricted to N . If g is any smooth function on M , the Jacobi bracket between the restrictions of f and g to N is given by

$$(4.4) \quad [f|_N, g|_N]_N = [f, g] + \sum_{i,j} [G_i, g] A_{ij} [G_j, f],$$

where A_{ij} is the inverse of the matrix $([G_i, G_j])$. In particular, if either Y_f or Y_g is tangent to N , then $[f|_N, g|_N]_N = [f, g]$.

PROOF. (a) From the conditions $[1, G_j] = 0|_N$, $j = 1, \dots, 2k$ and (4.2), it easily follows that dG_1, \dots, dG_{2k} are linearly independent semi-basic forms on N , and that the Reeb vector field Z is tangent to N . Hence N is a submanifold transverse to \mathcal{H}_M .

Let $\mathcal{H}_N = \{\xi \in T_x N \mid \alpha(\xi) = 0\}$. We need to prove that $d\alpha$ is non-degenerate on \mathcal{H}_N , i.e. that \mathcal{H}_N is a symplectic subbundle of \mathcal{H}_M . Owing to $\dim \mathcal{H}_M = 2n$, $\dim \mathcal{H}_N = 2n - 2k$, we obtain that the dimension of the symplectic orthogonal to \mathcal{H}_N within \mathcal{H}_M equals $2k$.

Since dG_1, \dots, dG_{2k} are independent semi-basic forms on N , we have that $Z, Y_{G_1}, \dots, Y_{G_{2k}}$, i.e., $\hat{Y}_{G_1}, \dots, \hat{Y}_{G_{2k}}$ are linearly independent vector fields on N . Whence, from

$$d\alpha(\hat{Y}_{G_j}, X) = -dG_j(X) = 0|_N, \quad X \in \mathcal{H}_N,$$

we see that $\text{orth}_{\mathcal{H}_M} \mathcal{H}_N$ is spanned by $\hat{Y}_{G_1}, \dots, \hat{Y}_{G_{2k}}$. The relations

$$d\alpha(\hat{Y}_{G_j}, \hat{Y}_{G_i}) = d\alpha(Y_{G_j}, Y_{G_i}) = [G_j, G_i]$$

and (4.2) yield that $\text{orth}_{\mathcal{H}_M} \mathcal{H}_N$ is symplectic, hence \mathcal{H}_N is symplectic as well.

Since Z is tangent to N , it is obvious that $Z|_N$ is the Reeb vector field of $(N, j^* \alpha)$. Also, note that we have the following decompositions of $T_x M$ at $x \in N$:

$$(4.5) \quad T_x M = \mathcal{Z}_x \oplus \mathcal{H}_{M_x} = \mathcal{Z}_x \oplus \mathcal{H}_{N_x} \oplus \text{orth}_{\mathcal{H}_M} \mathcal{H}_{N_x} = T_x N \oplus \text{orth}_{\mathcal{H}_M} \mathcal{H}_{N_x}.$$

(b) According to (1.5), we can write the equations for multipliers (4.3) in terms of the Jacobi brackets

$$(4.6) \quad \sum_i \lambda_i [G_i, G_j] = [f, G_j], \quad j = 1, \dots, 2k,$$

where we used $G_j|_N \equiv 0$, $j = 1, \dots, 2k$.

Thus, (4.3) has a unique solution on N ,

$$\lambda_i(f) = \sum_j A_{ij} [f, G_j],$$

determining the projection W_f of Y_f to TN with respect (4.5). From

$$i_{W_f} d\alpha(\xi) = -df(\xi) + \sum_{j=1}^{2k} \lambda_j dG_j(\xi) = -df(\xi), \quad \xi \in T_x N$$

we obtain that the contact Hamiltonian vector field of $f|_N$ reads

$$\begin{aligned} Y_f^* &= \hat{W}_f + fZ \\ &= Y_f - \sum_i \lambda_i(f) Y_{G_i} - (\alpha(Y_f) - \sum_i \alpha(\lambda_i(f) Y_{G_i})) Z + fZ \\ &= Y_f - \sum_i \lambda_i(f) Y_{G_i} - (f - \sum_i \lambda_i(f) G_i) Z + fZ \\ &= Y_f - \sum_i \lambda_i(f) Y_{G_i}. \end{aligned}$$

Finally, let g be a smooth function on M . Then

$$\begin{aligned} [f|_N, g|_N]_N &= Y_f^*(g) - gZ(f) \\ &= Y_f(g) - gZ(f) - \sum_i \lambda_i(f) Y_{G_i}(g) \\ &= [f, g] - \sum_i \lambda_i(f) [G_i, g] \\ &= [f, g] + \sum_{i,j} [G_i, g] A_{ij} [G_j, f]. \end{aligned}$$

□

Now, let us consider a general situation, when the Reeb vector field Z of (M, α) is not tangent to a contact co-oriented submanifold $(N, j^*\alpha)$. Then, analogous to (4.5), we have the decompositions of $T_x M$ at the points $x \in N$:

$$(4.7) \quad \begin{aligned} T_x M &= \mathcal{Z}_x \oplus \mathcal{H}_{M_x} = \mathcal{Z}_x \oplus \mathcal{H}_{N_x} \oplus \text{orth}_{\mathcal{H}_M} \mathcal{H}_{N_x} \\ &= \mathcal{Z}^* \oplus \mathcal{H}_{N_x} \oplus \text{orth}_{\mathcal{H}_M} \mathcal{H}_{N_x} = T_x N \oplus \text{orth}_{\mathcal{H}_M} \mathcal{H}_{N_x}, \end{aligned}$$

where \mathcal{Z}^* is a linear subbundle of TN spanned by the Reeb vector field Z^* of $(N, j^*\alpha)$. For a given smooth function f on M , the contact Hamiltonian vector field of $f|_N$ reads

$$Y_f^* = \hat{W}_f + fZ^* = W_f - \alpha(W_f)Z + fZ^*,$$

where W_f is the projection to TN of Y_f with respect to the decomposition (4.7).

If N is given by (4.1), then we have the same condition (4.2) on the constraints G_1, \dots, G_{2k} and the same equations (4.6) determining the multipliers $\lambda_1, \dots, \lambda_{2k}$, while the expression for the Jacobi bracket (4.4) is different:

$$\begin{aligned} [f|_N, g|_N]_N &= Y_f^*(g) - gZ^*(f) \\ &= Y_f(g) - \sum_i \lambda_i(f)Y_{G_i}(g) - (f - \sum_i \lambda_i(f)G_i)Z(g) + fZ^*(g) - gZ^*(f) \\ &= [f, g] + gZ(f) - fZ(g) + fZ^*(g) - gZ^*(f) - \sum_i \lambda_i(f)Y_{G_i}(g). \end{aligned}$$

4.1. Integrability of the Reeb flows. Suppose that the Reeb flow

$$\dot{x} = Y_1 = Z$$

on a contact manifold (M^{2n+1}, α) is integrable by means of integrals $f_1, f_2, \dots, f_{2n-r}$ satisfying (1.7) and let M_{reg} denote a open set foliated on invariant pre-isotropic tori of dimension $r+1$. In addition, assume that the system is subjected to the constraints (4.1) satisfying conditions of Theorem 4.1.

Since the Reeb vector field on N is the restriction $Z|_N$, it is clear that, in the case $M_{reg} \cap N$ is an open dense set of N , a generic Reeb trajectory on N will be quasi-periodic. However, as the example with the Reeb flow on a Brieskorn manifold suggests (see the next section), the restricted flow not need to be integrable by means of integrals obtained by the restrictions of $f_1, f_2, \dots, f_{2n-r}$ to N .

Denote the foliation of M_{reg} on invariant pre-isotropic tori by \mathcal{F} . Then \mathcal{F} is a α -complete pre-isotropic foliation and we have the following flag of foliations

$$(4.8) \quad \mathcal{G} = \mathcal{F} \cap \mathcal{H}_M \subset \mathcal{F} = \mathcal{Z} \oplus \mathcal{G} \subset \mathcal{E} = \mathcal{Z} \oplus \text{orth}_{\mathcal{H}_M} \mathcal{G}$$

Let \mathcal{F}_N be the foliation given by the level sets $f_i(x) = c_i$, $i = 1, \dots, 2n-r$, $x \in N_{reg}$, that is $\mathcal{F}_N = \mathcal{F}|_N \cap TN$ (if necessary, in order to obtain a foliation, we restrict functions to some open dense set of N_{reg}). Further, let $\mathcal{G}_N = \mathcal{F}_N \cap \mathcal{H}_N = \mathcal{G}|_N \cap \mathcal{H}_N$ be an isotropic foliation of N_{reg} . Then the Reeb flow is integrable by means of integrals $f_1|_N, \dots, f_{2n-r}|_N$ if the distribution

$$\mathcal{E}_N = \mathcal{Z} \oplus \text{orth}_{\mathcal{H}_N} \mathcal{G}_N$$

is a foliation (see [18]). We have

$$\begin{aligned}\mathcal{E}_N &= \mathcal{Z} \oplus \text{orth}_{\mathcal{H}_N} \mathcal{G}_N = \mathcal{Z} \oplus (\text{orth}_{\mathcal{H}_M}(\mathcal{G} \cap \mathcal{H}_N) \cap \mathcal{H}_N) \\ &= \mathcal{Z} \oplus ((\text{orth}_{\mathcal{H}_M} \mathcal{G} + \text{orth}_{\mathcal{H}_M} \mathcal{H}_N) \cap \mathcal{H}_N) \\ &= \mathcal{Z} \oplus \text{pr}_{\mathcal{H}_N}(\text{orth}_{\mathcal{H}_M} \mathcal{G}) = \text{pr}_{TN}(\mathcal{Z} \oplus \text{orth}_{\mathcal{H}_M} \mathcal{G}) = \text{pr}_{TN} \mathcal{E},\end{aligned}$$

where $\text{pr}_{\mathcal{H}_N}$ and pr_{TN} denote the projections with respect to the decompositions of TM given in (4.5).

In particular, if $\mathcal{F}|_N$ is tangent to N ($\mathcal{F}_N = \mathcal{F}|_N$), then \mathcal{F}_N is α -complete preisotropic foliation and the Reeb flow on N is integrable. Indeed, in this case \mathcal{E}_N is an integrable distribution since $\mathcal{G}|_N \subset \mathcal{H}_N$ implies

$$\text{pr}_{\mathcal{H}_N}(\text{orth}_{\mathcal{H}_M} \mathcal{G}) = \text{orth}_{\mathcal{H}_M} \mathcal{G} \cap \mathcal{H}_N \quad \text{and} \quad \text{pr}_{TN} \mathcal{E} = \mathcal{E} \cap TN.$$

For example, if the generic Reeb trajectories in N_{reg} are everywhere dense over the $r + 1$ -dimensional isotropic tori then it is obvious that $\mathcal{F}_N = \mathcal{F}|_N$.

5. Contact flows on Brieskorn manifolds

In Section 2 we started from a system of identical Harmonic oscillators, and gave a simple construction of a contact hypersurface in $(\mathbb{R}^{2n+2}(q, p), dp \wedge dq)$ with a integrable Reeb flow associated to the Gefand–Cetin system on $u(n + 1)$.

In this section we are going into a different direction. We also start from a system of harmonic oscillators (2.11), but then change our contact manifold imposing the constraints in the well known construction of the contact structure on Brieskorn manifolds [23, 10].

5.1. Complete contact integrability on a sphere. In this section we use the identification $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, $z_j = x_j + iy_j$, ($j = 0, \dots, n$). Let

$$F(z, \bar{z}) = \sum_{j=0}^n |z_j|^2.$$

Then the unit sphere S^{2n+1} may be expressed as the level surface $F = 1$.

In [33], the following basic examples of K -contact manifolds are given:

PROPOSITION 5.1. *Let a_j ($j = 0, \dots, n$) be positive real numbers and*

$$\alpha = \frac{i}{8} \sum_{j=0}^n a_j (z_j d\bar{z}_j - \bar{z}_j dz_j) = \frac{1}{4} \sum_{j=0}^n a_j (x_j dy_j - y_j dx_j).$$

Then (S^{2n+1}, α) is a co-oriented contact manifold with the Reeb vector field

$$(5.1) \quad Z = 4i \sum_{j=0}^n \frac{1}{a_j} \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

Note that the coordinate transformation

$$(5.2) \quad p_i = \sqrt{\frac{a_i}{2}} x_i, \quad q_i = \sqrt{\frac{a_i}{2}} y_i, \quad i = 0, \dots, n$$

transforms the contact ellipsoid $(E_{1/4}, \alpha_0)$ to the contact sphere (S^{2n+1}, α) . Now, the contact integrability of the Reeb flow on (S^{2n+1}, α) directly follows from Theorem 2.1. However, it is instructive to have a direct proof as well.

PROPOSITION 5.2. *The flow induced by the Reeb vector field (5.1) is completely contact integrable on (S^{2n+1}, α) and the functions*

$$(5.3) \quad f_j(z) = |z_j|^2, \quad j = 0, \dots, n$$

are its commuting integrals.

PROOF. Let

$$Y_j = \frac{4i}{a_j} \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right), \quad j = 0, \dots, n.$$

From

$$Zf_j = df_j(Z) = (\bar{z}_j dz_j + z_j d\bar{z}_j)(Z) = \frac{4i}{a_j} (\bar{z}_j z_j - z_j \bar{z}_j) = 0$$

and $dF(Y_j) = \frac{4i}{a_j} (\bar{z}_j z_j - z_j \bar{z}_j) = 0$, we conclude that f_j are integrals of (5.1) and that Y_j are tangent to S^{2n+1} . Since

$$i_{Y_j} d\alpha = -(z_j d\bar{z}_j + \bar{z}_j dz_j) = -df_j$$

and $\alpha(Y_j) = f_j$, it follows that Y_j are the contact Hamiltonian vector fields of the functions f_j . Whence, from (1.5), we obtain

$$[f_j, f_k] = Y_j f_k = df_k(Y_j) = 0, \quad j, k = 0, \dots, n.$$

Obviously, the functions f_j , $j = 0, \dots, n$ are independent on the open dense subset $U = \{(z_0, \dots, z_n) \mid z_0 \cdot \dots \cdot z_n \neq 0\}$ of \mathbb{C}^{n+1} , while the restrictions of f_j , $j = 1, \dots, n$ to S^{2n+1} are independent on $S^{2n+1} \cap U$. Formally, in the polar coordinates

$$(5.4) \quad z_j = r_j e^{i\varphi_j}, \quad j = 0, \dots, n,$$

we have $dF \wedge df_1 \wedge \dots \wedge df_n = 2^{n+1} r_0 r_1 \dots r_n dr_0 \wedge dr_1 \wedge \dots \wedge dr_n \neq 0|_U$. Therefore, $df_1 \wedge \dots \wedge df_n \neq 0|_{S^{2n+1} \cap U}$, considered as a n -form on S^{2n+1} . \square

Consider a $(n+1)$ -dimensional invariant torus

$$(5.5) \quad T_c : \quad f_0 = c_0^2, \quad \dots, \quad f_n = c_n^2, \quad c_0^2 + c_1^2 + \dots + c_n^2 = 1$$

that lay in $S^{2n+1} \cap U$ ($c_0, \dots, c_n > 0$). After a coordinate change (5.4), the Reeb dynamics on T_c is given by

$$\dot{\varphi}_j = \omega_j = \frac{4}{a_j}, \quad j = 0, \dots, n.$$

If $\omega_0, \omega_1, \dots, \omega_n$ (that is, a_0, a_1, \dots, a_n) are independent over \mathbb{Q} , then the trajectories are everywhere dense over T_c . In general, if among $\omega_0, \omega_1, \dots, \omega_n$ we have r independent relations

$$\rho_k : \quad m_0^k \omega_0 + \dots + m_n^k \omega_n = 0, \quad m_i^k \in \mathbb{Z}, \quad k = 1, \dots, r,$$

the dimension $n + 1 - r$ of the closure of trajectories laying on T_c is equal to the rank of (S^{2n+1}, α) considered as a K -contact manifold [33].

5.2. Reduction to Brieskorn manifolds. Now, let a_j ($j = 0, \dots, n$) be positive integers. Then the rank of (S^{2n+1}, α) is equal to 1, and the Reeb flow induces a circle action.

Let $G(z) = \sum_{j=0}^n z_j^{a_j}$. The set

$$B = \{z \in \mathbb{C}^{n+1} : F(z, \bar{z}) = 1, G(z) = 0\}$$

is known as Brieskorn manifold and (B, α) is a co-oriented contact manifold with the Reeb vector field (5.1) (see [23]). Therefore, obviously, the system is integrable in a noncommutative sense.

Note that

$$Z(G) = 4i \cdot G,$$

implying

$$(5.6) \quad [G_1, 1] = [G_2, 1] = 0|_B,$$

as well as $[G_1, G_2] \neq 0|_B$ (see the equation (5.7) below). Here, we denoted

$$G_1(z, \bar{z}) = \frac{1}{2} \sum_{j=0}^n (z_j^{a_j} + \bar{z}_j^{a_j}) = \Re(G), \quad G_2(z, \bar{z}) = \frac{1}{2i} \sum_{j=0}^n (z_j^{a_j} - \bar{z}_j^{a_j}) = \Im(G).$$

Therefore, the construction presented in Theorem 4.1 provides an alternative proof of the fact that (B, α) is a co-oriented contact manifold.

In what follows we shall describe the Jacobi brackets $[f, g]_B$ within the Lie algebra of integrals of the Reeb flow.

LEMMA 5.1. *Define*

$$V_1 = 2i \sum_{j=0}^n \left(\bar{z}_j^{a_j-1} \frac{\partial}{\partial z_j} - z_j^{a_j-1} \frac{\partial}{\partial \bar{z}_j} \right), \quad V_2 = -2 \sum_{j=0}^n \left(\bar{z}_j^{a_j-1} \frac{\partial}{\partial z_j} + z_j^{a_j-1} \frac{\partial}{\partial \bar{z}_j} \right).$$

Then $V_1(z), V_2(z)$ are tangent to S^{2n+1} for all $z \in B$. The restrictions to B of the contact Hamiltonian vector fields Y_{G_j} on (S^{2n+1}, α) are given by $\hat{V}_j|_B$, $j = 1, 2$.

PROOF. We have

$$dF(V_1) = 2i \sum_{j=0}^n (\bar{z}_j^{a_j} - z_j^{a_j}) = 0|_B, \quad dF(V_2) = -2 \sum_{j=0}^n (\bar{z}_j^{a_j} + z_j^{a_j}) = 0|_B,$$

which proves the first assertion. Now,

$$\begin{aligned}
 i_{Y_{G_1}} d\alpha = i_{V_1} d\alpha &= \frac{i}{4} \sum_{j=0}^n a_j dz_j \wedge d\bar{z}_j(V_1, \cdot) \\
 &= \frac{i}{4} \sum_{j=0}^n a_j (dz_j(V_1) d\bar{z}_j - d\bar{z}_j(V_1) dz_j) \\
 &= -\frac{1}{2} \sum_{j=0}^n a_j (\bar{z}_j^{a_j-1} d\bar{z}_j + z_j^{a_j-1} dz_j) \\
 &= -dG_1,
 \end{aligned}$$

and similarly for dG_2 . \square

THEOREM 5.1. *Let f and g be integrals of the Reeb vector field (5.1). Then*

$$[f, g]_B = [f, g] + \frac{df(V_2)dg(V_1) - df(V_1)dg(V_2)}{\mu},$$

where $[\cdot, \cdot]_B$ is the Jacobi bracket on (B, α) , $[\cdot, \cdot]$ is the Jacobi bracket on (S^{2n+1}, α) and

$$\mu = 2 \sum_{j=0}^n a_j |z_j|^{2(a_j-1)} = 2 \sum_{j=0}^n a_j f_j^{a_j-1} \neq 0.$$

PROOF. Let Y_f, Y_{G_1}, Y_{G_2} be the contact Hamiltonian vector fields on (S^{2n+1}, α) and

$$W_f = Y_f - \lambda_1 Y_{G_1} - \lambda_2 Y_{G_2}.$$

Taking into account (1.5), (5.6), and Lemma 5.1, we have

$$(5.7) \quad [G_1, G_2] = Y_{G_1} G_2 = dG_2(Y_{G_1}) = dG_2(V_1) = \mu \neq 0|_B.$$

Theorem 4.1 implies that the system

$$(5.8) \quad dG_1(W_f) = dG_2(W_f) = 0,$$

has a unique solution and $Y_f^* = W_f$ is the contact Hamiltonian vector field on (B, α) . In particular, (5.8) leads to

$$(5.9) \quad \lambda_1 = \frac{dG_2(Y_f)}{\mu}, \quad \lambda_2 = -\frac{dG_1(Y_f)}{\mu}$$

and since $Z(f) = Z(g) = 0$, we get

$$\begin{aligned}
 [f, g]_B &= [f, g] + \left(-\frac{dG_2(Y_f)}{\mu} Y_{G_1} + \frac{dG_1(Y_f)}{\mu} Y_{G_2} \right) g \\
 &= [f, g] + \left(-\frac{dG_2(Y_f)}{\mu} V_1 + \frac{dG_1(Y_f)}{\mu} V_2 \right) g.
 \end{aligned}$$

Finally, from

$$dG_j(Y_f) = -d\alpha(Y_{G_j}, Y_f) = -d\alpha(V_j, Y_f) = d\alpha(Y_f, V_j) = -df(V_j), \quad j = 1, 2,$$

we conclude the proof. \square

COROLLARY 5.1. *The integrals (5.3) do not commute on (B, α) .*

PROOF. From $df_j(V_1) = 2i(\bar{z}_j^{a_j} - z_j^{a_j})$, $df_j(V_2) = -2(\bar{z}_j^{a_j} + z_j^{a_j})$ and $[f_j, f_k] = 0$, we obtain

$$\begin{aligned} [f_j, f_k]_B &= \frac{4i}{\mu} \left[(\bar{z}_j^{a_j} - z_j^{a_j})(\bar{z}_k^{a_k} + z_k^{a_k}) - (\bar{z}_j^{a_j} + z_j^{a_j})(\bar{z}_k^{a_k} - z_k^{a_k}) \right] \\ &= \frac{8i}{\mu} [\bar{z}_j^{a_j} z_k^{a_k} - z_j^{a_j} \bar{z}_k^{a_k}] \neq 0, \end{aligned}$$

for $j \neq k$. \square

PROPOSITION 5.3. *The complete noncommutative set of integrals of the Reeb flow on the Brieskorn manifold (B, α) is given by f_j , $[f_j, f_k]_B$, $j, k = 0, \dots, n$.*

PROOF. The integrals f_j and

$$f_{j,k} = \frac{\mu}{8} [f_j, f_k]_B = i [\bar{z}_j^{a_j} z_k^{a_k} - z_j^{a_j} \bar{z}_k^{a_k}]$$

provide a complete set of noncommuting integrals for the Reeb flow on S^{2n+1} (their level sets are the Reeb circles). Therefore, after their restrictions to the Brieskorn manifold B , we get a complete set of integrals for a Reeb flow on B .

Indeed, consider a torus (5.5) and a coordinate change (5.4). The integrals $f_{j,k}$, restricted to T_c ,

$$f_{j,k}|_{T_c} = i c_j^{a_j} c_k^{a_k} \left[e^{i(a_k \varphi_k - a_j \varphi_j)} - e^{-i(a_k \varphi_k - a_j \varphi_j)} \right],$$

correspond to the rational relations

$$\rho_{j,k} : m_j \omega_j + m_k \omega_k = 0, \quad m_j = a_j, \quad m_k = a_k.$$

Since among $\rho_{j,k}$ we have n independent relations, among $f_{j,k}|_{T_c}$ we have n independent functions. Thus, among $f_j, f_{j,k}$ we have $2n$ independent functions on S^{2n+1} , that is $2n - 2$ independent functions on B . \square

In [30], Ustilovsky studied the Brieskorn manifolds (B_p, α_p) with

$$(5.10) \quad a_0 = p, \quad a_1 = \dots = a_n = 2,$$

where $n = 2m + 1$ and $p \equiv \pm 1 \pmod{8}$. The manifold B_p is diffeomorphic to a standard sphere S^{4m+1} [11], and as shown in [30], for $p_1 \neq p_2$, the contact structures $\mathcal{H}_{p_1} = \ker \alpha_{p_1}$ and $\mathcal{H}_{p_2} = \ker \alpha_{p_2}$ are not isomorphic. The proof is based on the study of periodic trajectories of the Reeb flow of the perturbed contact form $\frac{1}{H} \alpha_p$, which is equal to the contact flow

$$(5.11) \quad \dot{z} = Y_H^*$$

on (B_p, α_p) , where

$$\begin{aligned} H &= F + \sum_{j=1}^m \epsilon_j g_j, \quad 0 < \epsilon_j < 1, \quad j = 1, \dots, m, \\ g_j &= i(\bar{z}_{2j} z_{2j+1} - z_{2j} \bar{z}_{2j+1}) = 2(y_{2j} x_{2j+1} - y_{2j+1} x_{2j}). \end{aligned}$$

From a point of view of integrability, we can consider the contact flow of H as an integrable perturbation of the Reeb flow.

PROPOSITION 5.4. *The contact flow (5.11) is completely integrable in a non-commutative sense. Generic invariant pre-isotropic tori are of dimension $m + 1$, spanned by the Reeb flow and the contact flows of integrals g_1, \dots, g_m .*

PROOF. Since f_i, g_j are integrals of the system of harmonic oscillators with conditions (5.10), after the coordinate transformation (5.2), from Theorem 2.1, we obtain the following commuting relations on (S^{4m+3}, α_p) :

$$\begin{aligned} [g_i, g_j] &= \{g_j, g_i\}|_{E_{\frac{1}{4}}} = 0, \\ [g_j, 1] &= \{H_0, g_j\}|_{E_{\frac{1}{4}}} = 0, \\ [f_0, g_j] &= \{g_j, f_0\}|_{E_{\frac{1}{4}}} = 0, \\ [f_1, g_j] &= \{g_j, f_1\}|_{E_{\frac{1}{4}}} = 0, \\ [h_i, g_j] &= \{g_j, h_i\}|_{E_{\frac{1}{4}}} = 0, \end{aligned}$$

where $h_j = f_{2j} + f_{2j+1}$, $j = 1, \dots, m$.

Next, since

$$\begin{aligned} dg_j(V_1) &= i(z_{2j+1}d\bar{z}_{2j} + \bar{z}_{2j}dz_{2j+1} - z_{2j+1}d\bar{z}_{2j} - \bar{z}_{2j}dz_{2j+1})(V_1) \\ &= -2(-z_{2j+1}z_{2j} + \bar{z}_{2j}\bar{z}_{2j+1} + z_{2j+1}z_{2j} - \bar{z}_{2j}\bar{z}_{2j+1}) = 0, \\ dg_j(V_2) &= i(z_{2j+1}d\bar{z}_{2j} + \bar{z}_{2j}dz_{2j+1} - z_{2j+1}d\bar{z}_{2j} - \bar{z}_{2j}dz_{2j+1})(V_2) \\ &= -2i(z_{2j+1}z_{2j} + \bar{z}_{2j}\bar{z}_{2j+1} - z_{2j+1}z_{2j} - \bar{z}_{2j}\bar{z}_{2j+1}) = 0, \end{aligned}$$

from Theorem 5.1 we get the commuting relations on (B_p, α_p) as well:

$$\begin{aligned} [g_i, g_j]_{B_p} &= 0, \quad [f_0, g_j]_{B_p} = 0, \quad [f_1, g_j]_{B_p} = 0, \\ [h_i, g_j]_{B_p} &= 0, \quad [g_j, 1]_{B_p} = 0. \end{aligned}$$

Within the set of integrals of the Reeb flow, the Jacobi bracket of two integrals of the system (5.11) is both the integral of (5.11) and of the Reeb flow. Thus, we have a set of integrals

$$(5.12) \quad g_j, \quad h_j = f_{2j} + f_{2j+1}, \quad f_0, \quad f_1, \quad [f_0, h_j]_{B_p}, \quad [f_1, h_j]_{B_p}, \quad [h_i, h_j]_{B_p},$$

that commute with g_k , for all $i, j, k = 1, \dots, m$.

For the noncommutative integrability, it remains to note that among the integrals (5.12) we have $3m$ independent functions, including g_1, \dots, g_m . Indeed,

$$(5.13) \quad dG_1 \wedge dG_2 \wedge dF \wedge dg_1 \wedge \dots \wedge dg_m \wedge dh_1 \wedge \dots \wedge dh_m \wedge dq_1 \wedge \dots \wedge dq_m \neq 0$$

holds on an open dense subset of B_p . Here, since μ is an integral of the system, instead of $[f_1, h_j]_{B_p}$ we consider as in Proposition 5.3 the integrals

$$q_j = \frac{\mu}{8}[f_1, h_j]_{B_p} = i[\bar{z}_1^2 z_{2j}^2 - z_1^2 \bar{z}_{2j}^2] + i[\bar{z}_1^2 z_{2j+1}^2 - z_1^2 \bar{z}_{2j+1}^2], \quad j = 1, \dots, m.$$

The relation (5.13) can be verified by straightforward calculations in the polar coordinates (5.4). \square

6. Note on K -contact manifolds

The integrability of the Reeb flow on a sphere (Proposition 5.2) is a particular case of the integrability of the Reeb flows on compact K -contact manifolds.

Let (M, α) be a $(2n + 1)$ -dimensional co-oriented contact manifold. Then $d\alpha$ induces a symplectic structure on the contact distribution $\mathcal{H} = \ker \alpha$. In this situation, there exist a positive definite metric $g_{\mathcal{H}}$ and an almost complex structure J on \mathcal{H} , such that

$$g_{\mathcal{H}}(X, Y) = d\alpha(X, JY), \quad g_{\mathcal{H}}(JX, JY) = g_{\mathcal{H}}(X, Y),$$

for all horizontal vector fields X and Y .

The metric $g := g_{\mathcal{H}} \oplus (\alpha \otimes \alpha)$ on M is called an *adapted metric* to the contact form α . If there exists an adapted metric g such that the Reeb vector field is a Killing vector field

$$\mathcal{L}_Z g = 0,$$

then we call (M, α, g) a *K -contact manifold*. For example, the Sasakian manifolds have a natural K -contact structure (e.g., see [10]).

The following statement is a simple modification of Proposition 2.1 in Yamazaki [33].

PROPOSITION 6.1. *The Reeb flow on a compact K -contact manifold (M, α, g) is (noncommutative) contact integrable.*

PROOF. The Reeb flow $\varphi(t)$ is an one-parametric subgroup in a compact Lie group G of isometries of (M, g) . It follows that the closure of $\{\varphi(t) \mid t \in \mathbb{R}\}$ in G is a torus \mathbb{T}^{r+1} for some integer r . Thus, from $\varphi(t)^*\alpha = \alpha$, we have $s^*\alpha = \alpha$ for all $s \in \mathbb{T}^{r+1}$ and (M, α) is a \mathbb{T}^{r+1} -contact manifold (the number $r + 1$ is called the *rank* of (M, α, g) [33, 10]).

Let ξ be in the Lie algebra \mathfrak{t}^{r+1} of \mathbb{T}^{r+1} . It induces a contact vector field X_{ξ} on M with a Hamiltonian $f_{\xi} = \alpha(X_{\xi})$ ($\mathbf{f} : M \rightarrow (\mathfrak{t}^{r+1})^*$, $\mathbf{f}(\xi) := f_{\xi}$ is known as a contact momentum mapping [21, 10]). Take a base ξ_0, \dots, ξ_r of \mathfrak{t}^{r+1} such that $Z = X_{\xi_0}$, that is $f_{\xi_0} \equiv 1$. Since \mathbb{T}^{r+1} is Abelian, we have $[X_{\xi_i}, X_{\xi_j}] = 0$ and $[f_{\xi_i}, f_{\xi_j}] = 0$, $i, j = 0, \dots, n$.

In a neighborhood of a generic point, the \mathbb{T}^{r+1} -action defines a foliation \mathcal{F} on invariant $r + 1$ -dimensional tori. In particular, according to (1.5), we have $d\alpha(\hat{X}_{\xi_i}, \hat{X}_{\xi_j}) = 0$, so the leaves of \mathcal{F} are pre-isotropic and $r \leq n$. Let f_{r+1}, \dots, f_{2n-r} be integrals of \mathcal{F} defined on an open dense set of M . Since $\mathcal{L}_{X_{\xi_j}}(f_i) = 0$, we have $[f_i, 1] = 0$, $[f_i, f_{\xi_j}] = 0$, $i = r + 1, \dots, 2n - r$, $j = 1, \dots, r$, which completes the proof. \square

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